Theorem 3 (Poole 3.7): Let A be a $n \times n$ matrix that is invertible. Then for any $\mathbf{b}$ in $\mathbb{R}^n$, the vector $A^{-1}\mathbf{b}$ is the *unique* solution to the linear system $A\mathbf{x} = \mathbf{b}$.

Proof: There are two things to show.

1. Show that $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

$A\vec{x} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I_n\vec{b} = \vec{b}$

2. Show that $\mathbf{x} = A^{-1}\mathbf{b}$ is the *only* solution to $A\mathbf{x} = \mathbf{b}$.

Suppose $A\vec{x} = \vec{b}$

$\vec{x} = I_n\vec{x} = (A^{-1}A)\vec{x} = A^{-1}(Ax) = A^{-1}\vec{b}$

Example 3: Find all solutions to the linear system

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{4}$$

Hint: We showed in example 1 that $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is invertible with inverse $A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

By theorem 3, the unique solution is given by

$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Check)

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ✓